Inverse Problems for Parabolic Partial Differential Equations

1 Origin of the Proposal

Inverse Problems are problems where causes for a desired or an observed effect are to be determined. They lie at the heart of scientific inquiry and technological development. Applications include a number of medical as well as other imaging techniques, location of oil and mineral deposits in the earth’s substructure, creation of astrophysical images from telescope data, finding cracks and interfaces within materials, refractive optics, deconvolution problems in physics, inverse heat conduction problems with application in steel processing, shape optimization, airport security, industrial process monitoring, model identification in growth processes, parameter identification in partial differential equations and modelling in the life sciences.

Historically, the term inverse problems came into common use in the middle of the 20th century, mainly due to the ideas of A.N. Tikhonov and M.M. Lavrentjev. At that period research of inverse problems was really considered as investigation in an inverse direction with respect to classical boundary value problems for known differential equations of mathematical physics. As a rule, inverse problems turn out to be incorrect in a classical sense, that is, they belong to the set of ill-posed problems. This formed was a base for opponents to inverse problems.

In particular, a famous Russian mathematician S.L. Sobolev once told that the incorrect problems just appeared due to wrong settings of the actual questions. Therefore he offered to set similar problems somehow differently. However that may be, the inverse problems have formed a wide scientific field, which is being investigated in many countries of the world. Moreover, the term inverse problems is applied rather generally including the cases, where the respective forward problems are absent. By the way, we do not take a risk to call all our problems by this term. Perhaps the idea expressed by S.L. Sobolev is embodied in direct mathematical modeling that means investigation of a mathematical model created specially for a given question. This approach works without any forward or inverse problems and allows one to decrease mathematical difficulties on the way. Simultaneously, it requires a high level of knowledge from a researcher in neighboring sciences, besides mathematics, needed for creation of an appropriate mathematical description of natural phenomena.

2 Objective

In many applications in technical sciences the identification of certain unknown parameters from measurement of another accessible quantity is of central importance. As for as the inverse problem for parabolic systems arising in mathematical biology and finance no satis-
factory result appeared. Therefore the study of stability analysis of the parabolic systems by deriving appropriate weighted Carleman estimates would be really an interesting proposal. In particular, in this proposed project, we propose to mainly concentrate on the stability results concerning the inverse problem of determining coefficients/source terms/memory kernel/initial conditions/boundary coefficients for parabolic partial differential equations in a bounded domain $\Omega \subset \mathbb{R}^n$ with the observation on a subdomain $\omega \subset \Omega$ or the boundary $\Sigma = \partial \Omega \times (0, T)$ and the Sobolev norm of certain partial derivatives of the solutions at a fixed positive time $\theta \in (0, T)$ over the whole spatial domain $\Omega$ and partial domain $\omega$.

In addition to that, the determination of unknown material properties based on boundary/surface measured data is one of the central and actual problems of partial differential equations (PDEs) in application to mathematical biology, mathematical finance, chemistry, computational material sciences. Mathematical modeling of these problems leads to inverse coefficient problems (ICP) for linear/nonlinear PDEs of various types. It is known from the theory of inverse problems that ICPs are most difficult in comparison with all other types of inverse problems. Moreover these problems are severely ill-posed even when the governing equations are linear which means that very close measured output data may correspond to quite different coefficients. In the literature, several approaches use the classical solution of the direct problem, which requires appropriate continuity or and differentiability properties of the unknown data. In practice, these data are obtained by physical measurements and may not be smooth functions. In this case the methods based on a classical solution cannot be applied. Using the theory of weak solution for PDEs [27] and quasi-solution approach [18], one can easily determine the unknown parameters in the given PDEs. Our aim is to study also the inverse problems governed by linear/nonlinear PDEs and related to the determination of unknown properties of coefficients based on boundary/surface measured data in the following sense:

**(ICP1)** determination of unknown coefficients/source of the PDEs from measured output data in the form of Dirichlet boundary conditions.

**(ICP2)** identification of unknown coefficients/source of the PDEs from measured output data in the form of Neumann boundary conditions.

**(ICP3)** identification of the unknown parameters of the given system from measured output data in the form the final time overdetermination data.

### 3 Methodology

In the rapidly developing field of inverse problems, there exist several methods that are used to study the stability results for an inverse problem for partial differential equations. For example, in the literature, Yamamoto [32] discusses the conditional stability for two dimensional heat equation with zero Neumann boundary condition using Tikhonov regularization. Apart from the method of regularization, we have multiplier technique, fixed
point technique, Hölder space approach, optimization technique, quasi-solution approach, Carleman estimates etc. for obtaining the stability estimate.

However, of all these methods used so far, the most effective in establishing such a result is the Carleman estimate method and quasi-solution approach. There have been great concern in the Carleman inequalities after the publication of the basic paper by Carleman in 1939. In particular, after the appearance of the fundamental results by Hörmander [14], this theory is one of the most developing areas of linear PDEs. Further the most common technique for identifying an unknown coefficient from some measured output is the method of output least squares (OLS) [19, 26, 28] (or quasi-solution approach, according to Ivanov [18], Tikhonov [31]).

Using this techniques the stability results of inverse problems for parabolic and hyperbolic differential equations have been obtained during the last two decades. Thus the Carleman estimate and quasi-solution approach are going to play a key role in this proposed work.

4 Significance of the Work

The importance of this proposal is to present the stability results for inverse problems for particular partial differential equations arising in population dynamics which have been intensively investigated for general parabolic equations during the last few decades. The main contributions to this area are due to A.L. Bukhgeim and M.V. Klibanov who popularized the method of global Carleman estimates in the context of inverse problems. Beginning with the work of Bukhgeim and Klibanov [4, 6, 21], Carleman estimates were also used to prove the uniqueness and stability results for multidimensional inverse problems with a single measurement. Imanuvilov and Yamamoto [15] discussed the Lipschitz stability results of determining a source term for general parabolic equations. For the determination of such a source term by a single measurement using Carleman estimates, we also refer to the interesting books by Bukhgeim [5] and Isakov [16], where conormal derivatives are taken as overdetermining data. The paper by Isakov [17] describes some general results on Carleman estimates of possible interest for mathematicians working on control theory or inverse problems for partial differential equations.

After these fundamental contributions to the study of inverse problems, there have been a lot of papers appearing in various directions. Yamamoto and Zou [33] investigated the simultaneous reconstruction of the initial temperature and heat radiative coefficient in a heat conduction system. Benabdallah et al. [3] established the uniqueness and stability results for both the discontinuous diffusion coefficient and the initial condition from a measurement of the solution on an arbitrary part of the boundary and at some arbitrary positive time for the heat equation. Recent paper by Klibanov [22] gives a brief review of the application of Carleman estimates to inverse problems for PDEs with respect to three fundamental issues, namely, uniqueness, stability, and numerical methods. The inverse problem of recovering co-
efficient from PDEs (particularly heat type equations) with variable coefficients by internal measurements arises naturally in geophysics. The recent book by Klibanov and Timonov [23] (and the cited references of therein) gives the insight that the weight functions associated with the Carleman estimates can also be used to construct the globally convergent numerical algorithms for coefficient inverse problems. Utilizing the Carleman estimates in the globally convergent convexification algorithm, Klibanov and Timonov [24] discussed the numerical experiments for two dimensional coefficient inverse problems of elliptic type and an inverse model problem of microwave imaging.

It should be emphasized that, to the best of our knowledge, as far as the inverse problem of parabolic system is concerned, there appeared few papers; see, Cristofol et al. [7] in which they discussed the simultaneous reconstruction of one coefficient and initial conditions for the reaction-diffusion system from the measurement of one solution over \((t_0, T) \times \omega\) and some measurement at fixed time \(T' \in (t_0, T)\) and the novelty of the paper by Cristofol et al. [8] is the identification of two coefficients with only one observation for a nonlinear reaction diffusion system. Since we have already done some work [1, 2, 29] in an inverse problems for parabolic systems using Carleman estimates, it would be more appropriate to carry out the research in inverse problems for the parabolic system arising in biology, chemistry and finance with the knowledge of these weighted estimates.

In quasi-solution approach, the unknown coefficient is chosen from an appropriate space \(K\) and the output, \(u(x, t, a)\), is computed by solving the direct problem. One defines an error functional, \(J[a] = \|u(x, t; a) - f\|_F\), comparing the computed output to the measured value, \(f\), in the norm of the output space, \(F\), and seeks to minimize the cost functional \(J[a]\) over the space \(K\).

Numerical implementation of these methods mainly uses gradient-type iteration algorithms. The iteration scheme of these algorithms has the form \(J(a(n)) = J(a^{(n-1)}) + \alpha_n J'(a^{(n)}), \quad n = 1, 2, 3, \cdots\), which uses the Fréchet gradient of the cost functional \(J_i(a)\) and the iteration parameter \(\alpha_n > 0\). The choice of this parameter defines various gradient methods, although in many situations estimation of the parameter \(\alpha_n > 0\) is a difficult problem. However, in the case of Lipschitz continuity of the gradient \(J'_i(a)\), this parameter can be estimated via the Lipschitz constant. The proposed approach is based on the introduction of appropriate (well-posed) adjoint problems with arbitrary data given at the final time \(t = T\), for each inverse problem. Note that the adjoint problem approach has first been introduced by DuChateau (see [9, 10]) for inverse coefficient problems related to linear and nonlinear parabolic equations and then developed in [11, 12]. Integral relationships, obtained here in the form of integral identities and corresponding to each inverse problem, permit one to establish relationships between the solutions of the direct problem and the corresponding adjoint problem. These integral relationships make the proposed approach more useful and capable of various mathematical and computational applications. In particular, using these relationships explicit formulae for the Fréchet gradients of the above cost functionals can be derived via the adjoint problems. These formulae show that, independently on the
given measured final data, both the inverse problems can be solved by the same gradient method. Moreover the strong results related to Lipschitz continuity of the gradients $J'_i(a)$ of the functionals permit one to estimate the iteration parameter $\alpha_n > 0$ in the case of both the inverse problems. This, in particular, shows that the gradient-type numerical methods, used for the inverse source problems for parabolic and hyperbolic equations, can also be used for the problems considered here. Note also that the approach presented here is a continuation of the methodology, given in [12]-[13], for inverse source problems for parabolic and hyperbolic equations.

5 Work Plan

The following are the short list of models to be consider in the proposed work:

- **Problem 1:** The study of pattern formation in bacterial colonies is of particular interest both from the biological and physical points of view. Consider a system consisting of bacterial cells and nutrient. Both cells and nutrient undergo diffusion while cells proliferate by consuming the nutrient. Let $u(x,t)$ be the population density of the cells at time $t$ and spatial position $x$ and the concentration of the nutrient be $v(x,t)$. Then $u$ and $v$ satisfy the following linearized system of coupled reaction diffusion equations:

\[
\begin{align*}
    u_t &= (d_1(x)u_x)_x + a(x)u + b(x)v + f(x,t), \quad (x,t) \in Q = \Omega \times (0,T], \\
    v_t &= (d_2(x)v_x)_x + c(x)u + d(x)v + g(x,t), \quad (x,t) \in Q,
\end{align*}
\]

with the initial and boundary conditions

\[
\begin{align*}
    u(x,0) &= 0, \quad v(x,0) = 0, \quad x \in I, \\
    u_x(0,t) &= 0, \quad u(l,t) = 0, \quad v_x(0,t) = 0, \quad v(l,t) = 0, \quad t \in (0,T],
\end{align*}
\]

where $\Omega = (0,l)$, which describe the evolution of the bacterium Bacillus subtilis, which grows on the surface of thin agar plates showing various patterns in response to environmental conditions such as the nutrient concentration, the solidity of an agar medium and temperature. The spatio-temporal patterns generated by the bacterium Bacillus subtilis were investigated by Kawasaki et al. in [20] using nonlinear reaction diffusion model. Here the strengths of the reaction $a(x), b(x), c(x)$ and $d(x)$ are assumed to be sufficiently smooth and shall be kept independent of time $t$; $d_1, d_2$ are the diffusion coefficients describing the bacterial movement and nutrient and the functions $f$ and $g$ are unknown source terms. The main concern of this work is to study reconstructing the source terms and the coefficients in the above linear reaction diffusion system as well as their nonlinear form with respect to the given additional Dirichlet/Neumann type output measured data and time dependent overspecified data.
**Problem 2:** We consider the Lotka-Volterra competition-diffusion system with three species in a bounded domain with Dirichlet boundary conditions and establish the stability result for the inverse problem consisting of retrieving two smooth coefficients $a(x)$ and $b(x)$ of the following system.

\[
\begin{align*}
\partial_t A_1 - \Delta A_1 + a_1 A_1 + a_2 A_2 + a_3 A_3 &= a(x)f(A_1,A_2,A_3) \quad \text{in } Q, \\
\partial_t A_2 - \Delta A_2 + a_4 A_2 + a_5 A_3 + a_6 A_1 &= b(x)g(A_1,A_2,A_3) \quad \text{in } Q, \\
\partial_t A_3 - \Delta A_3 + a_7 A_3 + a_8 A_1 + a_9 A_2 &= 0 \quad \text{in } Q,
\end{align*}
\]

with suitable boundary conditions, where $Q = \Omega \times (0,T)$, $\Sigma = \partial \Omega \times (0,T)$ and $\Omega$ is an open bounded subset of $\mathbb{R}^n$, $n \leq 3$, with boundary $\partial \Omega$ of class $C^2$. The coefficients $a_i \in L^\infty(Q)$, $i = 1,2,\cdots,9$. The nonlinear functions $f$ and $g$ satisfy generalized Lipschitz condition. The unknown coefficients $a(x)$ and $b(x)$ are assumed to be sufficiently smooth and shall be kept independent of time $t$.

**Problem 3:** In the theory of epidemics, a basic model for the description of the susceptible and infective populations is the so-called Kermack-McKendrick equations. When the effect of diffusion is taken into consideration, these equations, for $t > 0$, $x \in \Omega$, are given by

\[
\begin{align*}
y_t - \nabla \cdot (ay) + b_1 y &= F_1 - d_1(G(y))y, \\
z_t - \nabla \cdot (cy) + b_2 z &= F_2 - d_2(G(y))z,
\end{align*}
\]

with the Dirichlet boundary data $y = 0, z = 0$, where $y = y(x,t), z = z(x,t)$ respectively represent the susceptible and infective populations, $a = a(x), c = c(x)$ are the diffusion coefficients, $b_1, b_2$ are the reaction rate constants, $d_1(x), d_2(x)$ reaction coefficients and $F_1, F_2$ are possible external sources. The functional $G(y)$ is continuous in the given domain.

During the last few years, the inverse problems for parabolic type differential equations have been intensively developed by many mathematical researchers. Here we mention some of the directions of this development related to our problem, which will be studied in our project.

- Reconstruction of both the coefficients of nonlinear functions $d_1(x)$ and $d_2(x)$.

- Simultaneous reconstruction of the following combinations

  * both $a(x)$ and $d_2(x)$ or both $b(x)$ and $d_1(x)$

  * all the coefficients $a(x), b(x), d_1(x)$ and $d_2(x)$

As far as we know, no paper is available in the literature to study the identification problems for the above said combination for any parabolic system.

**Problem 4:** Suppose that a given single population species is free to move in an open and bounded habitat by $p(x,t,a)$ the density of individuals of age $a \in [0,a_1], a_1 \in$
at time $t \geq 0$ and location $x \in \Omega$. From the law of Nernst, one can obtain the following system:

$$p_t + p_a - k\Delta p + \mu p = f(x, t, a) \quad (x, t, a) \in \Omega \times (0, T) \times (0, a_1).$$

Most appropriate boundary conditions for this problem are $p(x, t, a) = 0, \ (x, t, a) \in \Sigma_T = \partial \Omega \times (0, T) \times (0, a_1)$, which describes the case of a completely inhospitable boundary. When there is no migration across the boundary, we consider the homogeneous Neumann boundary condition and when there is a migration of population across the boundary, which is proportional to the density $p$ on the boundary $\partial \Omega$, we consider mixed boundary condition.

The birth process is described by the renewal law

$$p(x, t, 0) = \int_0^{a_1} \beta(x, t, a)p(x, t, a)da, \ (x, t) \in Q,$$

where $\beta$ is the fertility rate and gives the proportion of newborn population at moment $t$ and location $x$, with parents of age $a$. The initial density of individuals is given by

$$p(x, 0, a) = p_0(x, a), \ (x, a) \in \Omega \times (0, a_1).$$

**Problem 5:** The simplest reaction-diffusion models for cyclic populations involve two interacting species, with densities $u$ and $v$ say, at different trophic levels

$$\begin{aligned}
  u_t &= \Delta u + au + bf(u, v) \quad (x, t) \in Q, \\
  v_t &= \Delta v + cv + dg(u, v) \quad (x, t) \in Q, \\
  u(x, \theta) &= u_\theta, \ v(x, \theta) = v_\theta \quad x \in \Omega, \\
  u(x, t) &= h_1(x, t), \ v(x, t) = h_2(x, t) \quad (x, t) \in \Sigma,
\end{aligned}$$

where $u$ and $v$ may be predator and prey, host and parasite, herbivore and grazer, etc. The parameters $a$ and $c$ are the respective intrinsic growth rates. The nonlinear functions are globally Lipschitz continuous.
References


